

Davenport constant of the multiplicative semigroup of the ring $\mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$

Guoqing Wang^{a*} Weidong Gao^{b†}

^aDepartment of Mathematics, Tianjin Polytechnic University, Tianjin, 300387, P. R. China

^bCenter for Combinatorics, LPMC-TJKLC, Nankai University, Tianjin 300071, P. R. China

Abstract

Given a finite commutative semigroup \mathcal{S} (written additively), denoted by $D(\mathcal{S})$ the Davenport constant of \mathcal{S} , namely the least positive integer ℓ such that for any ℓ elements $s_1, \dots, s_\ell \in \mathcal{S}$ there exists a set $I \subseteq [1, \ell]$ for which $\sum_{i \in I} s_i = \sum_{i=1}^{\ell} s_i$.

Then, for any integers $r \geq 1, n_1, \dots, n_r > 1$, let $R = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$ be the direct sum of these r residue class rings $\mathbb{Z}_{n_1}, \dots, \mathbb{Z}_{n_r}$. Moreover, let \mathcal{S}_R be the multiplicative semigroup of the ring R , and $U(\mathcal{S}_R)$ the group of units of \mathcal{S}_R . In this paper, we prove that

$$D(U(\mathcal{S}_R)) + P_2 \leq D(\mathcal{S}_R) \leq D(U(\mathcal{S}_R)) + \delta,$$

where $P_2 = \#\{i \in [1, r] : 2 \nmid n_i\}$ and $\delta = \#\{i \in [1, r] : 2 \mid n_i\}$. This corrects our previous published wrong result on this problem.

Key Words: Davenport constant; Multiplicative semigroups; Residue class rings

1 Introduction

The Davenport constant of any finite abelian group G , denoted $D(G)$, is the smallest positive integer ℓ such that, every sequence T of elements in G of length at least ℓ contains a nonempty subsequence T' with the sum of all terms of T' equaling the identity element of G . Though

*Email: gqwang1979@aliyun.com

†Email : wdgaol963@aliyun.com

attributed to H. Davenport who proposed the study of this constant in 1965, K. Rogers [3] in 1963 pioneered the investigation of a combinatorial invariant associated with an arbitrary finite abelian group G . The Davenport constant is a central concept of zero-sum theory and has been investigated by many researchers in the scope of finite abelian groups.

In 2008, the two authors of this manuscript formulated the definition of the Davenport constant of finite commutative semigroups. Subsequently, some related additive results are obtained in the setting of semigroups (see [1, 4–6, 8, 9]).

Definition A. ([7]) *Let S be a finite commutative semigroup. Let T be a sequence of terms from the semigroup S . We call T reducible if T contains a proper subsequence T' ($T' \neq T$) such that the sum of all terms of T' equals the sum of all terms of T . Define the Davenport constant of the semigroup S , denoted $D(S)$, to be the smallest $\ell \in \mathbb{N}$ such that every sequence T of length at least ℓ of elements in S is reducible.*

In 2006, A. Geroldinger and F. Halter-Koch had introduced another combinatorial invariant, which they denoted by $d(S)$, (see Definition 2.8.12 in [2]), and now is called the small Davenport constant of S after [1]; this is closely related to the Davenport constant of S , as it is known from Proposition 1.2 in [1] that $D(S) = d(S) + 1$ for any finite commutative semigroup S .

It is embarrassing that the following obtained result on the Davenport constant of finite commutative semigroups was observed to be incorrect when the ring R is even in some cases.

Theorem B. ([7]) *For integers $r \geq 1$, $n_1, \dots, n_r > 1$, let $R = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_r}$. Let \mathcal{S}_R be the multiplicative semigroup of the ring R and $U(\mathcal{S}_R)$ be the group of units of \mathcal{S}_R . Then*

$$D(\mathcal{S}_R) = D(U(\mathcal{S}_R)) + P_2,$$

where $P_2 = \#\{i \in [1, r] : 2 \parallel n_i\}$.

It is high time to correct this mistake. In this paper, we shall prove the following result by employing a different method with the previous one used in [7].

Theorem 1.1. *For integers $r \geq 1$, $n_1, \dots, n_r > 1$, let $R = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_r}$. Let \mathcal{S}_R be the multiplicative semigroup of the ring R and $U(\mathcal{S}_R)$ be the group of units of \mathcal{S}_R . Then*

$$D(U(\mathcal{S}_R)) + P_2 \leq D(\mathcal{S}_R) \leq D(U(\mathcal{S}_R)) + \delta,$$

where $P_2 = \#\{i \in [1, r] : 2 \parallel n_i\}$ and $\delta = \#\{i \in [1, r] : 2 \mid n_i\}$.

2 The preliminaries

• In the rest of this manuscript, we shall always admit that \mathcal{S} is a *unitary* finite commutative semigroup.

The operation on \mathcal{S} is denoted by $+$. The identity element of \mathcal{S} , denoted $0_{\mathcal{S}}$, is the unique element e of \mathcal{S} such that $e+a = a$ for every $a \in \mathcal{S}$. Let $U(\mathcal{S}) = \{a \in \mathcal{S} : a+a' = 0_{\mathcal{S}} \text{ for some } a' \in \mathcal{S}\}$ be the group of units of \mathcal{S} . For any element $c \in \mathcal{S}$, let

$$\text{St}(c) = \{a \in U(\mathcal{S}) : a + c = c\}$$

denote the stabilizer of c in the group $U(\mathcal{S})$. Green's preorder on the semigroup \mathcal{S} , denoted $\leq_{\mathcal{H}}$, is defined by

$$a \leq_{\mathcal{H}} b \Leftrightarrow a = b \text{ or } a = b + c \text{ for some } c \in \mathcal{S}.$$

Green's congruence on \mathcal{S} , denoted \mathcal{H} , is defined by:

$$a \mathcal{H} b \Leftrightarrow a \leq_{\mathcal{H}} b \text{ and } b \leq_{\mathcal{H}} a.$$

We write $a <_{\mathcal{H}} b$ to mean that $a \leq_{\mathcal{H}} b$ but $a \mathcal{H} b$ does not hold.

A sequence T of \mathcal{S} is denoted by $T = a_1 a_2 \cdots a_{\ell} = \coprod_{a \in \mathcal{S}} a^{[v_a(T)]}$, where $[v_a(T)]$ means that the element a occurs $v_a(T)$ times in the sequence T . By \cdot we denote the operation to join sequences. By $|T|$ we denote the length of the sequence, i.e., $|T| = \sum_{a \in \mathcal{S}} v_a(T) = \ell$. Let T_1, T_2 be two sequences of \mathcal{S} . We call T_2 a subsequence of T_1 if $v_a(T_2) \leq v_a(T_1)$ for every element $a \in \mathcal{S}$, denoted by $T_2 \mid T_1$. In particular, if $T_2 \neq T_1$, we call T_2 a *proper* subsequence of T_1 , and write $T_3 = T_1 T_2^{[-1]}$ to mean the unique subsequence of T_1 with $T_2 \cdot T_3 = T_1$. Let ε be the *empty sequence*. In particular, the empty sequence ε is a proper subsequence of any nonempty sequence. If T is a nonempty sequence, then we let $\sigma(T) = \sum_{a \in \mathcal{S}} [v_a(T)]a$. We also define $\sigma(\varepsilon) = 0_{\mathcal{S}}$. We say that T is *reducible* if $\sigma(T') = \sigma(T)$ for some proper subsequence T' of T .

In what follows, we denote

$$R = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_r}$$

where $r \geq 1, n_1, \dots, n_r > 1$ are integers, and denote \mathcal{S}_R to be the multiplicative semigroup of the ring R . For any element $\mathbf{a} \in \mathcal{S}_R$, we denote $\theta_{\mathbf{a}} = (a_1, \dots, a_r) \in [1, n_1] \times \cdots \times [1, n_r]$ be the unique r -tuple of integers such that $(\bar{a}_1, \dots, \bar{a}_r)$ is the corresponding form of the element \mathbf{a} in the ring R . Let $\kappa_i(\theta_{\mathbf{a}}) = a_i$ where $i \in [1, r]$. We remark that, since the operation of the semigroup \mathcal{S}_R is always denoted by $+$, for any elements \mathbf{a}, \mathbf{b} and \mathbf{c} of \mathcal{S}_R , $\mathbf{a} + \mathbf{b} = \mathbf{c}$ holds in \mathcal{S}_R if and only if $\kappa_i(\theta_{\mathbf{a}}) * \kappa_i(\theta_{\mathbf{b}}) \equiv \kappa_i(\theta_{\mathbf{c}}) \pmod{n_i}$ for all $i = 1, 2, \dots, r$.

Here, the following two lemmas are necessary.

Lemma 2.1. ([2], Lemma 6.1.3) *Let G be a finite abelian group, and let H be a subgroup of G . Then, $D(G) \geq D(G/H) + D(H) - 1$.*

• For any prime p and any integer $n \neq 0$, let $\text{pot}_p(n)$ be largest integer k such that p^k divides n .

Lemma 2.2. *Let a and b be two elements of \mathcal{S}_R . Then the following conclusions hold:*

- (i) *If $a \leq_{\mathcal{H}} b$, then $\gcd(\kappa_i(\theta_b), n_i) \mid \gcd(\kappa_i(\theta_a), n_i)$ for each $i \in [1, r]$, and $\text{St}(b) \subseteq \text{St}(a)$;*
- (ii) *$a \mathcal{H} b$ if and only if $\gcd(\kappa_i(\theta_b), n_i) = \gcd(\kappa_i(\theta_a), n_i)$ for each $i \in [1, r]$;*
- (iii) *Suppose $a <_{\mathcal{H}} b$. If there exists some index $t \in [1, r]$ such that*

$$\text{pot}_p(\gcd(\kappa_t(\theta_b), n_t)) < \text{pot}_p(\gcd(\kappa_t(\theta_a), n_t))$$

for some prime $p > 2$, or

$$\text{pot}_2(\gcd(\kappa_t(\theta_b), n_t)) < \text{pot}_2(\gcd(\kappa_t(\theta_a), n_t)) < \text{pot}_2(n_t),$$

then $\text{St}(b) \subsetneq \text{St}(a)$.

Proof. (i) Note that $a \leq_{\mathcal{H}} b$ implies $a = b + c$ for some $c \in \mathcal{S}_R$ since \mathcal{S}_R is unitary, and equivalently, $\kappa_i(\theta_a) \equiv \kappa_i(\theta_b) * \kappa_i(\theta_c) \pmod{n_i}$ for each $i \in [1, r]$. Then Conclusion (i) follows from a routine verification.

(ii) Assume $\gcd(\kappa_i(\theta_b), n_i) = \gcd(\kappa_i(\theta_a), n_i)$ for each $i \in [1, r]$. It follows that there exist integers $c_i, c'_i \in [0, n_i - 1]$ such that $\kappa_i(\theta_a) * c_i \equiv \kappa_i(\theta_b) \pmod{n_i}$ and $\kappa_i(\theta_b) * c'_i \equiv \kappa_i(\theta_a) \pmod{n_i}$, where $i \in [1, r]$. Take elements $c, c' \in \mathcal{S}_R$ such that $\kappa_i(\theta_c) = c_i$ and $\kappa_i(\theta_{c'}) = c'_i$ for each $i \in [1, r]$. It follows that $a + c = b$ and $b + c' = a$, and so $a \mathcal{H} b$. While, the converse follows from Conclusion (i) immediately.

(iii) Let q be the largest prime with

$$\text{pot}_q(\gcd(\kappa_t(\theta_b), n_t)) < \text{pot}_q(\gcd(\kappa_t(\theta_a), n_t)). \quad (1)$$

Let

$$\alpha = \text{pot}_q(\gcd(\kappa_t(\theta_a), n_t)). \quad (2)$$

Take an element $d \in \mathcal{S}_R$ with

$$\kappa_i(\theta_d) = 1 \text{ for each } i \in [1, r] \setminus \{t\}, \quad (3)$$

and with

$$\kappa_t(\theta_d) \equiv \begin{cases} 2\frac{n_t}{q^\alpha} + 1 \pmod{n_t} & \text{if } q > 2 \text{ and } \gcd(2\frac{n_t}{q^\alpha} + 1, n_t) = 1; \\ \frac{n_t}{q^\alpha} + 1 \pmod{n_t} & \text{if otherwise.} \end{cases} \quad (4)$$

Then we have the following.

Assertion A. $d \in U(\mathcal{S}_R)$.

Proof of Assertion A. By (3), it suffices to establish that $\gcd(\kappa_t(\theta_d), n_t) = 1$. If $q = 2$, then $\kappa_t(\theta_d) \equiv \frac{n_t}{q^\alpha} + 1 \pmod{n_t}$, and so $\gcd(\kappa_t(\theta_d), n_t) = 1$ follows from the hypothesis, $\text{pot}_2(\gcd(\kappa_t(\theta_b), n_t)) < \text{pot}_2(\gcd(\kappa_t(\theta_a), n_t)) < \text{pot}_2(n_t)$, immediately. Assume

$$q > 2.$$

We show that $\gcd(\frac{n_t}{q^\alpha} + 1, n_t) = 1$ or $\gcd(2\frac{n_t}{q^\alpha} + 1, n_t) = 1$. Suppose to the contrary that $\gcd(\frac{n_t}{q^\alpha} + 1, n_t) > 1$ and $\gcd(2\frac{n_t}{q^\alpha} + 1, n_t) > 1$. Since $w \nmid \frac{n_t}{q^\alpha} + 1$ and $w \nmid 2\frac{n_t}{q^\alpha} + 1$ for every prime divisor w of n_t with $w \neq q$, it follows that $q \mid \frac{n_t}{q^\alpha} + 1$ and $q \mid 2\frac{n_t}{q^\alpha} + 1$, which implies that $q \mid 2(\frac{n_t}{q^\alpha} + 1) - (2\frac{n_t}{q^\alpha} + 1) = 1$, which is absurd. This proves Assertion A. \square

By (1), (2), (3) and (4), we check that $\kappa_i(\theta_d) * \kappa_i(\theta_a) \equiv \kappa_i(\theta_a) \pmod{n_i}$ for each $i \in [1, r]$, and $\kappa_t(\theta_d) * \kappa_t(\theta_b) \not\equiv \kappa_t(\theta_b) \pmod{n_t}$. That is, $d + a = a$ and $d + b \neq b$. Combined with Assertion A, we have that $d \in \text{St}(a) \setminus \text{St}(b)$ proved. This completes the proof of Lemma 2.2. \square

3 The proof of Theorem 1.1

Proof of Theorem 1.1. We first prove that $D(\mathcal{S}_R) \geq D(U(\mathcal{S}_R)) + P_2$. Assume without loss of generality that

$$\{i \in [1, r] : 2 \parallel n_i\} = [1, P_2].$$

Take an irreducible sequence A of terms from $U(\mathcal{S}_R)$ of length $D(U(\mathcal{S}_R)) - 1$. Let

$$B = A \cdot \prod_{i=1}^{P_2} b_i \quad (5)$$

where $\theta_{b_i} = (1, \dots, 1, 2, 1, \dots, 1)$ with 2 appears at the i -th location for each $i \in [1, P_2]$. Now we show that B is an irreducible sequence. Suppose to the contrary that B contains a proper subsequence B' with

$$\sigma(B') = \sigma(B). \quad (6)$$

Since $\sigma(B') \mathcal{H} \sigma(B)$, it follows from Lemma 2.2 (ii) that $\prod_{i=1}^{P_2} b_i \mid B'$, say

$$B' = A' \cdot \prod_{i=1}^{P_2} b_i \quad (7)$$

where A' is a proper subsequence of A . By (5), (6) and (7), we derive that

$$2\kappa_i(\theta_{\sigma(A')}) \equiv 2\kappa_i(\theta_{\sigma(A)}) \pmod{n_i} \text{ for each } i \in [1, P_2] \quad (8)$$

and

$$\kappa_j(\theta_{\sigma(A')}) \equiv \kappa_j(\theta_{\sigma(A)}) \pmod{n_j} \text{ for each } j \in [P_2 + 1, r]. \quad (9)$$

Since $\kappa_i(\theta_{\sigma(A')})$ and $\kappa_i(\theta_{\sigma(A)})$ are odd, it follows from (8) that $\kappa_i(\theta_{\sigma(A')}) \equiv \kappa_i(\theta_{\sigma(A)}) \pmod{n_i}$, where $i \in [1, P_2]$. Combined with (9), we have that $\sigma(A') = \sigma(A)$, which is a contradiction. This proves that B is irreducible, and so $D(\mathcal{S}_R) \geq |B| + 1 = D(U(\mathcal{S}_R)) + P_2$. Now it remains to show that $D(\mathcal{S}_R) \leq D(U(\mathcal{S}_R)) + \delta$.

Let $T = a_1 \cdot a_2 \cdots a_\ell$ be an arbitrary sequence of term from the semigroup \mathcal{S}_R of length $\ell = D(U(\mathcal{S}_R)) + \delta$. It suffices to show that T contains a *proper* subsequence T' with $\sigma(T') = \sigma(T)$. Take a shortest subsequence V of T such that

$$\sigma(V) \mathcal{H} \sigma(T). \quad (10)$$

We may assume without loss of generality that

$$V = a_1 \cdot a_2 \cdots a_t \text{ where } t \in [0, \ell].$$

If $t = 0$, i.e., $V = \varepsilon$, then $\sigma(V) = 0_{\mathcal{S}_R}$, which implies that $\sigma(T) \in U(\mathcal{S}_R)$ by (10). It follows that T is a sequence of terms from the group $U(\mathcal{S}_R)$ and of length $|T| = D(U(\mathcal{S}_R)) + \delta \geq D(U(\mathcal{S}_R))$, and thus, T is reducible, we are done. Hence, we assume that

$$t > 0.$$

By the minimality of $|V|$, we derive that

$$0_{\mathcal{S}_R} >_{\mathcal{H}} a_1 >_{\mathcal{H}} (a_1 + a_2) >_{\mathcal{H}} \cdots >_{\mathcal{H}} \sum_{i=1}^t a_i. \quad (11)$$

Recall that an empty sum of elements of \mathcal{S}_R is taken equal to $0_{\mathcal{S}_R}$. Denote $K_i = \text{St}(\sum_{j=1}^i a_j)$ where $i \in [0, t]$. Note that K_i is a subgroup of $U(\mathcal{S}_R)$ for each $i \in [0, t]$. Combined with (11) and Lemma 2.2 (i), we have that

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots \subseteq K_t. \quad (12)$$

Moreover, we have the following.

Assertion B. There exists a subset $M \subseteq [0, t-1]$ with $|M| \geq t-\delta$ such that $K_i \subsetneq K_{i+1}$ for each $i \in M$.

Proof of Assertion B. Let $v \in [0, t-1]$ be an arbitrary index with $K_v = K_{v+1}$. We shall apply Lemma 2.2 by taking $a = \sum_{i=1}^{v+1} a_i$ and $b = \sum_{i=1}^v a_i$. Since $a <_{\mathcal{H}} b$, it follows from Lemma 2.2 that $\gcd(\kappa_j(\theta_b), n_j) \mid \gcd(\kappa_j(\theta_a), n_j)$ for each $j \in [1, r]$, and moreover,

$$\text{pot}_2(\gcd(\kappa_w(\theta_b), n_w)) < \text{pot}_2(\gcd(\kappa_w(\theta_a), n_w)) = \text{pot}_2(n_w)$$

for some $w \in [1, r]$. By the arbitrariness of v , we have Assertion B proved. \square

For each $m \in M$, since $\frac{U(\mathcal{S}_R)}{K_{m+1}} \cong \frac{U(\mathcal{S}_R)/K_m}{K_{m+1}/K_m}$ and $D(K_{m+1}/K_m) \geq 2$, it follows from Lemma 2.1 that

$$\begin{aligned} D(U(\mathcal{S}_R)/K_{m+1}) &= D\left(\frac{U(\mathcal{S}_R)/K_m}{K_{m+1}/K_m}\right) \\ &\leq D(U(\mathcal{S}_R)/K_m) - (D(K_{m+1}/K_m) - 1) \\ &\leq D(U(\mathcal{S}_R)/K_m) - 1. \end{aligned} \quad (13)$$

By (12), (13) and Assertion B, we conclude that

$$\begin{aligned} 1 \leq D(U(\mathcal{S}_R)/K_t) &\leq D(U(\mathcal{S}_R)/K_0) - |M| \\ &\leq D(U(\mathcal{S}_R)) - (t - \delta) \\ &= (\ell - \delta) - (t - \delta) \\ &= \ell - t \\ &= |TV^{[-1]}|. \end{aligned} \quad (14)$$

By (10) and Lemma 2.2 (ii), we have

$$\gcd(\kappa_i(\theta_{\sigma(V)}), n_i) = \gcd(\kappa_i(\theta_{\sigma(T)}), n_i) \quad (15)$$

for each $i \in [1, r]$. Let

$$\mathcal{P}_i = \{p : p \text{ is prime with } \text{pot}_p(\gcd(\kappa_i(\theta_{\sigma(V)}), n_i)) = \text{pot}_p(n_i) > 0\}$$

where $i \in [1, r]$.

Let a be an arbitrary term of $TV^{[-1]}$. By (15), we have that for each $i \in [1, r]$,

$$q \nmid \kappa_i(\theta_a) \quad (16)$$

for any prime divisor q of n_i with $q \notin \mathcal{P}_i$. By the Chinese Remainder Theorem, we can choose integers $\tilde{a}_i \in [0, n_i - 1]$ such that

$$\tilde{a}_i \equiv 1 \pmod{p^{\text{pot}_p(n_i)}} \quad \text{for any prime } p \in \mathcal{P}_i \quad (17)$$

and

$$\tilde{a}_i \equiv \kappa_i(\theta_a) \pmod{q^{\text{pot}_q(n_i)}} \quad \text{for any prime divisor } q \text{ of } n_i \text{ with } q \notin \mathcal{P}_i, \quad (18)$$

where $i \in [1, r]$. Let \tilde{a} be the element of \mathcal{S}_R with $\kappa_i(\theta_{\tilde{a}}) = \tilde{a}_i$ for each $i \in [1, r]$. By (16), (17) and (18), we conclude that $\gcd(\kappa_i(\theta_{\tilde{a}}), n_i) = 1$ for each $i \in [1, r]$, i.e.,

$$\tilde{a} \in \text{U}(\mathcal{S}_R). \quad (19)$$

By (17) and (18), we conclude that

$$\tilde{a}_i * \kappa_i(\theta_{\sigma(V)}) \equiv 1 \cdot \kappa_i(\theta_{\sigma(V)}) \equiv 0 \equiv \kappa_i(\theta_a) * 0 \equiv \kappa_i(\theta_a) * \kappa_i(\theta_{\sigma(V)}) \pmod{p^{\text{pot}_p(n_i)}}$$

for any prime $p \in \mathcal{P}_i$, and that

$$\tilde{a}_i * \kappa_i(\theta_{\sigma(V)}) \equiv \kappa_i(\theta_a) * \kappa_i(\theta_{\sigma(V)}) \pmod{q^{\text{pot}_q(n_i)}}$$

for any prime divisor q of n_i with $q \notin \mathcal{P}_i$, that is,

$$\sigma(V) + \tilde{a} = \sigma(V) + a \quad \text{for each term } a \text{ of } TV^{[-1]}. \quad (20)$$

By (14), (19) and the arbitrariness of the element a above, we see that $\coprod_{a|TV^{[-1]}} \tilde{a}$ is a nonempty sequence of terms from $\text{U}(\mathcal{S}_R)$ of length $|\coprod_{a|TV^{[-1]}} \tilde{a}| = |TV^{[-1]}| \geq D(\text{U}(\mathcal{S}_R)/K_t)$. It follows that there exists a *nonempty* subsequence $W \mid TV^{[-1]}$ such that $\sigma(\coprod_{a|W} \tilde{a}) \in K_t$ which implies

$$\sigma(V) + \sigma(\coprod_{a|W} \tilde{a}) = \sigma(V). \quad (21)$$

By (20) and (21), we conclude that

$$\begin{aligned} \sigma(T) &= \sigma(TW^{[-1]}V^{[-1]}) + (\sigma(V) + \sigma(W)) \\ &= \sigma(TW^{[-1]}V^{[-1]}) + (\sigma(V) + \sigma(\coprod_{a|W} \tilde{a})) \\ &= \sigma(TW^{[-1]}V^{[-1]}) + \sigma(V) \\ &= \sigma(TW^{[-1]}), \end{aligned}$$

and $T' = TW^{[-1]}$ is the desired proper subsequence of T . This completes the proof of the theorem. \square

4 Concluding remarks

We remark that the upper bound in Theorem 1.1 could be reached in some cases. For example, take

$$R = \mathbb{Z}_8^{r_1} \oplus \mathbb{Z}_4^{r_2} \oplus \mathbb{Z}_2^{r_3}.$$

Note that $U(\mathcal{S}_R) \cong C_2^{2r_1+r_2}$ is the direct sum of $2r_1 + r_2$ cyclic groups of order two. It is well known that $D(U(\mathcal{S}_R)) = 2r_1 + r_2 + 1$. For each $i \in [1, r_1 + r_2 + r_3]$, we take $a_i \in \mathcal{S}_R$ with $\theta_{a_i} = (1, \dots, 1, 2, 1, \dots, 1)$ where the only 2 occurs at the i -th location. It is not hard to check that the sequence

$$T = \left(\prod_{i=1}^{r_1} a_i^{[3]} \right) \cdot \left(\prod_{j=r_1+1}^{r_1+r_2} a_j^{[2]} \right) \cdot \left(\prod_{k=r_1+r_2+1}^{r_1+r_2+r_3} a_k \right)$$

is an irreducible sequence in the semigroup \mathcal{S}_R . It follows that $D(\mathcal{S}_R) \geq |T| + 1 = 3r_1 + 2r_2 + r_3 + 1 = D(U(\mathcal{S}_R)) + r_1 + r_2 + r_3 = D(U(\mathcal{S}_R)) + \delta$, and thus,

$$D(\mathcal{S}_R) = D(U(\mathcal{S}_R)) + \delta.$$

However, it would be attractive to determine the precise value of $D(\mathcal{S}_R) - D(U(\mathcal{S}_R))$ when $R = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_r}$ for any integers $r \geq 1, n_1, \dots, n_r > 1$. Even, the following conjecture seems to be interesting.

Conjecture 4.1. *For integers $r \geq 1, n_1, \dots, n_r > 1$, let $R = \mathbb{Z}_{n_1} \oplus \dots \oplus \mathbb{Z}_{n_r}$. Then*

$$D(\mathcal{S}_R) - D(U(\mathcal{S}_R)) \leq \#\{i \in [1, r] : \text{pot}_2(n_i) \in [1, 3]\}.$$

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